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Cartesian monads on toposes

Peter Johnstone*

Department of Pure Mathematics, University of Cambridge, 16 Mill Lane, Cambridge, CB2 1SB, UK

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Dedicated to Peter Freyd on the occasion of his 60th birthday

Abstract

We consider two related questions: 'When can a cartesian functor between toposes be factored as an inverse image functor followed by a direct image functor?' and 'Does every cartesian monad on a topos $\mathscr{E}$ arise from a geometric morphism with codomain $\mathscr{E}$?'. The connection between the two questions is that the answer to the first, in the particular case when both the domain and codomain of the functor are the topos of sets, is 'if and only if it carries a monad structure'. We investigate the class of cartesian monads on the topos of sets, providing a new proof that they correspond bijectively to strongly zero-dimensional locales in this topos. By combining two old results on strong functors, we show that any cartesian monad on an arbitrary topos has a canonical extension to an indexed monad; this suffices to extend the arguments employed in the topos of sets to an arbitrary Boolean base topos, but we also show how they fail in non-Boolean toposes. For such toposes, the answer to the second question therefore remains unknown. © 1997 Elsevier Science B.V.

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0. Introduction

A geometric morphism of toposes $f : \mathscr{F} \to \mathscr{E}$, being an adjoint pair of functors, induces both a comonad $\mathbb{G}$ on the domain topos $\mathscr{F}$ and a monad $\mathbb{T}$ on $\mathscr{E}$. It is well known that the category of coalgebras for $\mathbb{G}$ is a topos, and provides the image of the morphism f ([6, 4.16]). A naive student might ask why no-one ever considers the category of algebras for $\mathbb{T}$. The short answer, of course, is that it is not a topos (unless $\mathbb{T}$ happens to be idempotent, cf. [6, 4.17]). But this cannot be the whole story: the

* E-mail: ptj@pmms.cam.ac.uk.

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functor part $f_* f^*$ of $\mathbb{T}$ preserves finite products, and so by the results of [12] we know that the category of $\mathbb{T}$-algebras is cartesian closed. And in the particular case when $\mathscr{E}$ is the topos of sets, the latter condition (plus non-degeneracy of $\mathbb{T}$) implies that the category of $\mathbb{T}$-algebras is 'very nearly' a topos, by the results of [10]: specifically, it is what we called in that paper the two-valued collapse of a topos. In particular, it follows that, for any such $\mathbb{T}$, there is a geometric morphism $f : \mathscr{F} \to \mathbf{Set}$ such that $\mathbb{T}$ is induced by the adjunction $(f^* \dashv f_*)$. (The topos $\mathscr{F}$ is not uniquely determined by $\mathbb{T}$; however, if we require it to be the topos whose two-valued collapse is the category of $\mathbb{T}$-algebras, then it is unique. Moreover, the toposes which arise in this way may be characterized as the toposes of sheaves on *strongly zero-dimensional locales*, as we shall see below.)

Shortly after completing [10], I began writing a paper in which I hoped to prove an analogous result for cartesian monads (that is, monads whose functor parts preserve finite limits – like Peter Freyd, I have long since abandoned the use of the doubly dead metaphor 'left exact' for functors preserving finite limits) on an arbitrary topos. (The paper was even cited in [9] as being 'in preparation'.) However, I found that I was unable to make the proof work except in the case when the base topos $\mathscr{E}$ was Boolean, so I set the paper aside. More recently, Mamuka Jibladze asked me a seemingly unrelated question about cartesian functors between toposes: when I tried to answer it, I realized that it was at least partly connected with the problem on cartesian monads that I had considered earlier. I still cannot answer the original question, but it now seems appropriate to set down what I do know about it, in the hope that it may stimulate further work on the problem.

Jibladze's question was also a 'naive dualization' of a well-known result of elementary topos theory: specifically, the result that every cartesian functor $T : \mathscr{E} \to \mathscr{F}$ between toposes can be factored as a direct image functor $p_* : \mathscr{E} \to \mathbf{Gl}(T)$ followed by an inverse image functor $q^* : \mathbf{Gl}(T) \to \mathscr{F}$, where $\mathbf{Gl}(T)$ is the topos obtained by Artin glueing along T (cf. [18, 3]). Jibladze asked: what can one say about those cartesian functors which can be factored as an inverse image functor followed by a direct image functor? We shall present some elementary observations about this question in Section 1, and show how it leads to the consideration of cartesian monads on toposes.

The rest of the paper is devoted to cartesian monads: in Section 2 we characterize the cartesian monads on $\mathbf{Set}$ in syntactic terms, and give a new proof (independent of that in [10]) that they correspond to strongly zero-dimensional locales. Section 3 is devoted to the proofs of two old but little-known results, neither of them due to the present author (one is essentially in Anders Kock's paper [12], and the other is an unpublished result of Bob Paré), which together imply that any cartesian monad on an arbitrary topos $\mathscr{E}$ automatically has the structure of an $\mathscr{E}$-indexed monad. This means, in particular, that it can be described in the syntactic terms that we are accustomed to use for monads on $\mathbf{Set}$. The final Section 4 is largely speculative: in it we indicate how one might hope to 'constructivize' the arguments of Section 2 and apply them in an arbitrary topos.

Overall, the paper is thus something of a 'rag-bag': a mixture of old and new results, of partial and complete results, and of my own results and ones which I have borrowed from other people. Nevertheless, I hope that these results will be found to be of some interest, and that they will not be considered wholly unworthy to be offered to Peter Freyd on the occasion of his sixtieth birthday.

1. Jibladze factorizations

Let $T : \mathscr{E} \to \mathscr{F}$ be a cartesian functor between toposes. By a *Jibladze factorization* of T, we shall mean a span

$$\mathscr{E} \xleftarrow{\ f\ } \mathscr{G} \xrightarrow{\ g\ } \mathscr{F} \tag{1}$$

of toposes and geometric morphisms such that T is naturally isomorphic to the composite $g_* f^*$. Jibladze's original question was whether every cartesian functor between toposes has a Jibladze factorization; but a moment's thought shows that the inclusion functor from finite sets to sets provides a counterexample. Slightly more generally, we have:

Lemma 1.1. *Let κ and λ be limit power cardinals with $\kappa < \lambda$. ('κ is a limit power cardinal' means that the category $\mathbf{Set}_\kappa$ of sets of cardinality less than κ is closed under powersets, and hence a topos; we allow the possibility that λ is ∞, the cardinality of the universe.) Then the inclusion functor $\mathbf{Set}_\kappa \to \mathbf{Set}_\lambda$ does not admit a Jibladze factorization.*

Proof. Suppose we have a span

$$\mathbf{Set}_\kappa \xleftarrow{\ f\ } \mathscr{G} \xrightarrow{\ g\ } \mathbf{Set}_\lambda\ .$$

Then since $\mathscr{G}$ is defined over $\mathbf{Set}_\lambda$, it has copowers indexed by sets of cardinality κ (cf. [6, 4.41]), and since it is defined over $\mathbf{Set}_\kappa$ its hom-sets all have cardinality less than κ. By a well-known argument due to Freyd [4, Exercise 3.D]), this forces $\mathscr{G}$ to be a preorder, and hence to be the degenerate topos; so the composite $g_* f^*$ is the constant functor with value 1. □

The argument in the proof of Lemma 1.1 works equally well if $\kappa > \lambda$, of course; but in this case it turns out that the only cartesian functor $\mathbf{Set}_\kappa \to \mathbf{Set}_\lambda$ is the one that admits a Jibladze factorization.

Lemma 1.2. *If κ and λ are cardinals with $\kappa > \lambda$, and $T : \mathbf{Set}_\kappa \to \mathbf{Set}_\lambda$ is a functor which preserves the terminal object and the equalizer diagram $\emptyset \to 1 \rightrightarrows 2$, then T is (isomorphic to) the constant functor with value* 1.

Proof. We use the extension of Freyd's argument by Isbell and Schanuel [5], which shows that any functor from a category with copowers of size λ to a category whose hom-sets have size less than λ must factor through a preorder (i.e. it must be constant on each hom-set). Any such functor must map split monomorphisms to isomorphisms; so we have $T1 \cong TA$ for each nonempty set A. If T preserves the equalizer in the statement of the lemma, then we also have $T\emptyset \cong T1$. □

What happens if $\kappa = \lambda$? To answer this, we shall assume for notational simplicity that $\kappa = \lambda = \infty$; but the argument which follows is not dependent on this assumption.

Lemma 1.3. *If a cartesian functor $T : \mathbf{Set} \to \mathbf{Set}$ admits a Jibladze factorization, then it carries a monad structure.*

Proof. Any topos admits at most one geometric morphism, up to isomorphism, to **Set** (see [6, 4.41]; so in the span diagram (1), we may as well assume that $f = g$. Hence $T \cong f_* f^*$ carries a monad structure. □

We shall see in the next section that the converse of Lemma 1.3 is true.

Of course, any cartesian functor $T : \mathbf{Set} \to \mathbf{Set}$ (indeed, any functor which preserves 1) admits a unique natural transformation $\eta : 1 \to T$, since the identity functor 1 is representable by the object 1; so the force of Lemma 1.3 is that T admits a natural transformation $\mu : T^2 \to T$ compatible in the appropriate sense with η. Which cartesian functors **Set** $\to$ **Set** have this property?

We recall that in [2] Andreas Blass showed that an arbitrary cartesian functor $T : \mathbf{Set} \to \mathbf{Set}$ can be expressed as a directed union of cartesian subfunctors which are isomorphic to *reduced power* functors, where the reduced power functor P_D corresponding to a filter D on a set I is defined by setting $P_D(X)$ to be the quotient of the set of all functions from members of D to X by the equivalence relation of agreeing on a set in D. (Except in the trivial case when D is the improper filter, we can actually restrict ourselves to functions $I \to X$.) Given the syntactic characterization of cartesian monads on **Set** which we shall establish in the next section, we shall be able to deduce

Proposition 1.4. *Suppose $T : \mathbf{Set} \to \mathbf{Set}$ is cartesian and carries a monad structure. Then every reduced power subfunctor of T is representable, i.e. corresponds to a principal filter. In particular, a reduced power functor carries a monad structure iff it corresponds to a principal filter.*

Before leaving this section, we remark that in seeking a general answer to Jibladze's question there is no loss of generality in restricting our attention to cartesian functors from a topos to itself. If $T : \mathscr{E} \to \mathscr{F}$ is any functor between toposes, let $\widetilde{T}$ denote the functor $\mathscr{E} \times \mathscr{F} \to \mathscr{E} \times \mathscr{F}$ defined on objects by $\widetilde{T}(A, B) = (1, TA)$. Clearly, $\widetilde{T}$ is cartesian iff T is; and we have

Lemma 1.5. *With the above notation, the functor T admits a Jibladze factorization iff $\widetilde{T}$ does.*

Proof. We have geometric morphisms

$$\mathscr{E} \xrightarrow{\;p\;} \mathscr{E} \times \mathscr{F} \xleftarrow{\;q\;} \mathscr{F}$$

whose inverse images are the projection functors, and whose direct images are the functors $A \mapsto (A, 1)$ and $B \mapsto (1, B)$. It is thus easy to see that if the span

$$\mathscr{E} \xleftarrow{\;f\;} \mathscr{G} \xrightarrow{\;g\;} \mathscr{F}$$

is a Jibladze factorization of T, then (pf, qg) is a Jibladze factorization of $\widetilde{T}$.

Conversely, suppose $\widetilde{T}$ has a Jibladze factorization

$$\mathscr{E} \times \mathscr{F} \xleftarrow{\;f\;} \mathscr{G} \xrightarrow{\;g\;} \mathscr{E} \times \mathscr{F}.$$

Since $\mathscr{E} \times \mathscr{F}$ is the coproduct of $\mathscr{E}$ and $\mathscr{F}$ in the 2-category **Top** of toposes and geometric morphisms [6, 4.21], and such coproducts are (disjoint and) stable under pullback, the morphisms f and g serve to decompose $\mathscr{G}$ as a four-fold coproduct, and $\widetilde{T}$ can be represented in the form $(A, B) \mapsto (T_1 A \times T_2 B, T_3 A \times T_4 B)$, where T_1–T_4 are the functors having Jibladze factorizations corresponding to the four summands of $\mathscr{G}$. Setting $B = 1$, we see that T_3 must be isomorphic to our original functor T (and the other three must be the constant functor with value 1). □

The argument in the second part of the proof of Lemma 1.5 can be generalized. If $\mathscr{E} = \prod_{i=1}^{n} \mathscr{E}_i$ and $\mathscr{F} = \prod_{j=1}^{m} \mathscr{F}_j$ are toposes which decompose as finite products (that is, as finite coproducts in **Top**), then any cartesian functor $T : \mathscr{E} \to \mathscr{F}$ (indeed, any functor which preserves finite products) determines and is determined by an $(n \times m)$ matrix of functors $T_{ij} : \mathscr{E}_i \to \mathscr{F}_j$: specifically, $T_{ij}(A)$ is the jth component of T applied to the n-tuple whose ith entry is A and whose other entries are 1, and $T(A_1, A_2, \ldots, A_n)$ is the m-tuple whose jth entry is $\prod_{i=1}^{n} T_{ij}(A_i)$, and we have:

Lemma 1.6. *With the notation developed above, the functor T has a Jibladze factorization iff each T_{ij} does.*

We omit the proof, which is similar to that of Lemma 1.5. We note, in particular, that (given the result already quoted from the next section) Lemma 1.6 enables us to characterize those functors $\mathbf{Set}^n \to \mathbf{Set}^m$ which admit Jibladze factorizations.

2. Hyperaffine theories and strongly zero-dimensional locales

We assume that the reader is familiar with the description of monads on **Set** in terms of (possibly infinitary) algebraic theories. If $\mathbb{T} = (T, \eta, \mu)$ is a monad on **Set**, we shall use the same symbol $\mathbb{T}$ for the algebraic theory which corresponds to it, and we shall identify the elements of TI, for a set I, with I-ary operations on $\mathbb{T}$-algebras.

We shall need to introduce some special notation. If α and β are operations of arities I and J, respectively, we shall write $\alpha * \beta$ for the $(I \times J)$-ary operation obtained by first applying α to each column of an $(I \times J)$ matrix of variables, and then applying β to the resulting J-tuple. Similarly, we shall write $\alpha \dagger \beta$ for the operation which first applies β to each row of the matrix, and then applies α. Of course, the assertion that the operations α and β commute with each other is just the $(I \times J)$-ary equation $\alpha * \beta = \alpha \dagger \beta$. If γ is an operation whose arity is a product $I \times J$, we shall write ${}_I\gamma$ (resp. γ_J) for the I-ary (resp. J-ary) operation obtained by applying γ to a matrix all of whose columns (resp. rows) are equal. And if α is an I-ary operation, we write α^d for the $(I \times I)$-ary operation obtained by applying α to the diagonal entries of an $(I \times I)$ matrix (and 'discarding' the other entries).

We recall that an operation α of a theory $\mathbb{T}$ is said to be *affine* if it satisfies the unary equation which says that the effect of applying α to an I-tuple whose members are all equal is the identity operation. Following [10], we shall call α *hyperaffine* if it is affine and satisfies the $(I \times I)$-ary equation $\alpha * \alpha = \alpha^d$. (These two conditions may be expressed diagrammatically by saying that, for every $\mathbb{T}$-algebra A, the diagrams

$$\begin{array}{ccc} A & \xrightarrow{A^{(I \to 1)}} & A^I \\ & \searrow^{1} & \downarrow{\scriptstyle \alpha_A} \\ & & A \end{array} \quad \text{and} \quad \begin{array}{ccc} A^{I \times I} & \xrightarrow{(\alpha_A)^I} & A^I \\ {\scriptstyle A^{\Delta}}\downarrow & & \downarrow{\scriptstyle \alpha_A} \\ A^I & \xrightarrow{\alpha_A} & A \end{array} \tag{2}$$

commute.) And we say $\mathbb{T}$ is an affine (resp. hyperaffine) theory if all its operations are affine (resp. hyperaffine).

We have the following characterization of hyperaffine algebraic theories.

Theorem 2.1. *For a monad* $\mathbb{T} = (T, \eta, \mu)$ *on* **Set**, *the following conditions are equivalent*:

(i) *The functor T preserves finite products.*

(ii) *The algebraic theory $\mathbb{T}$ is commutative, and the canonical symmetric monoidal closed structure on* $\mathbf{Set}^{\mathbb{T}}$ *is cartesian closed.*

(iii) *$\mathbb{T}$ is a hyperaffine theory.*

Proof. The equivalence of (ii) and (iii) was proved (under the hypothesis that $\mathbb{T}$ was commutative) in [10], and we shall not repeat the proof here. The equivalence of (i) and (iii) (and the fact that (i) implies (ii)) is essentially in Kock's paper [12], but since that paper deals with monads on arbitrary cartesian closed categories the proof may be hard to follow, and we shall give a more elementary proof here. We shall in fact show that (i) implies (iii) and the commutativity of $\mathbb{T}$, and that (iii) implies (i).

First suppose (i) holds. Since T preserves 1, the only unary operation of $\mathbb{T}$ is the identity, and so every operation of $\mathbb{T}$ must be affine. The assertion that T preserves binary products implies, in particular, that an $(I \times J)$-ary operation γ is uniquely determined by the pair $({}_I\gamma, \gamma_J)$; but if α and β are affine operations of arities I and J, respectively, then $\alpha * \beta$ and $\alpha \dagger \beta$ are both solutions of the equations ${}_I\gamma = \alpha$ and $\gamma_J = \beta$, and so they must be equal. Hence $\mathbb{T}$ is commutative. Similarly, given that α is affine, the operations $\alpha * \alpha$ and α^d are both solutions of ${}_I\gamma = \gamma_I = \alpha$, so they must be equal.

Conversely, if every operation of $\mathbb{T}$ is (hyper)affine, then $T1 = 1$ since the only unary operation is the identity. To show that T preserves binary products, we need to show that for any pair (α, β) there is a unique γ with ${}_I\gamma = \alpha$ and $\gamma_J = \beta$; but we have already observed that if α and β are affine then $\gamma = \alpha * \beta$ is a solution of these equations. For the uniqueness, we have simply to observe that if γ is an $(I \times J)$-ary hyperaffine operation, then it satisfies $\gamma = {}_I\gamma * \gamma_J$. □

Before proceeding further, we should remark that for a functor T defined on **Set** there is not a great deal of difference between the assertions 'T preserves finite products' and 'T is cartesian'. For a functor which preserves finite products will preserve all equalizers iff it preserves equalizers of coreflexive pairs; and, in **Set**, a coreflexive equalizer diagram $A \rightarrowtail B \rightrightarrows C$ can be given the structure of a split equalizer (and is therefore preserved by all functors), unless we have $A = \emptyset$ and $B \neq \emptyset$. So we are reduced to considering preservation of such equalizers. Further, if $T : \mathbf{Set} \to \mathbf{Set}$ preserves finite products, then since the projection $A \times \emptyset \to \emptyset$ is an isomorphism for all A we deduce that either $T\emptyset = \emptyset$ or $TA \cong 1$ for all A. In the latter case, T clearly preserves all limits; in the former, at least if T underlies a monad $\mathbb{T}$, it will preserve the particular equalizers mentioned above provided the two maps $1 \rightrightarrows 2$ have distinct images under T, i.e. provided the algebraic theory $\mathbb{T}$ is non-degenerate. Thus, the only monad on **Set** whose functor part preserves finite products but not finite limits is that corresponding to the degenerate theory with no constants, i.e. the monad whose functor part satisfies $T\emptyset = \emptyset$ and $TA = 1$ for all non-empty A.

Using the characterization provided by Theorem 2.1, we may now prove a remarkable property of cartesian monads on **Set**. Of course, any cartesian functor preserves finite intersections of subobjects, since these are simply pullbacks; but we have:

Lemma 2.2. *Let $\mathbb{T} = (T, \eta, \mu)$ be a monad on* **Set**, *such that T is cartesian. Then T preserves arbitrary intersections of subobjects.*

Proof. We have to show that, for any (non-empty) set I and any element α of TI, there is a unique smallest subset $I' \subseteq I$ such that α is in the image of $TI' \rightarrowtail TI$. But if we think of α as an I-ary operation on $\mathbb{T}$-algebras, then (since T preserves equalizers) the assertion that it lies in the image of this map is equivalent to saying that it satisfies the equation

$$\alpha(x_i \mid i \in I) = \alpha(x'_i \mid i \in I), \tag{3}$$

where the two I-tuples of variables satisfy $x_i = x'_i$ iff $i \in I'$. Let I_1 be the set of all those $i \in I$ such that α does not depend on its ith variable, i.e. it satisfies the above equation for $I' = I \setminus \{i\}$; and let $I_0 = I \setminus I_1$. We claim that α satisfies (3) for $I' = I_0$; it is clear that any I' for which (3) holds must satisfy $I \setminus I' \subseteq I_1$, so this suffices to prove the result.

To prove the claim, we use the hyperaffine identity for α. Consider the following two $(I \times I)$ matrices of variables (x_{ij}) and (y_{ij}): we set $x_{ij} = x_i$ for all j, and we set $y_{ij} = x'_i$ $(\neq x_i)$ if $i = j \in I_1$, otherwise $y_{ij} = x_i$. Clearly, applying α to any column of the first matrix has the same result as applying it to the corresponding column of the second matrix; so $\alpha * \alpha$ has the same effect on the two matrices. But if we apply α^d to the two matrices, we obtain the two sides of Eq. (3) for $I' = I_0$. □

Armed with this result, we may now give the proof of Proposition 1.4, which was postponed from Section 1.

Proof. As shown in [2], a reduced product subfunctor of a cartesian functor $T : \mathbf{Set} \to \mathbf{Set}$ corresponds to a pair (I, α) where I is a set and $\alpha \in TI$: the image of the induced natural transformation from the representable functor $\mathbf{Set}(I, -)$ to T is isomorphic to P_D, where D is the filter of those subsets $I' \subseteq I$ such that α is in the image of $TI' \rightarrowtail TI$. The cartesianness of T ensures that D is a filter; but we have just seen that, for a functor T which carries a monad structure, it is actually a principal filter, and so P_D is isomorphic to the functor $\mathbf{Set}(I_0, -)$ where I_0 is the least element of D. □

We now turn to proving the converse of Lemma 1.3. Suppose we happen to know that a particular cartesian monad $\mathbb{T}$ on **Set** is induced by the adjunction $(f^* \dashv f_*)$ for some **Set**-topos $f : \mathscr{E} \to \mathbf{Set}$. How much of $\mathscr{E}$ can we hope to recover from $\mathbb{T}$? First we note that if we have a connected geometric morphism $h : \mathscr{F} \to \mathscr{E}$ over **Set** (that is, one such that h^* is full and faithful), then $\mathscr{E}$ and $\mathscr{F}$ will give rise to the same monad on **Set**; so, by the factorization theorem of [7], we may as well assume that $\mathscr{E}$ is localic over **Set**, i.e. that it is the topos of sheaves on a locale. However, different localic toposes can still induce the same monad on **Set**; so we need to cut down still further.

Definition 2.3. We recall that a locale X is said to be *strongly zero-dimensional* if it is zero-dimensional (i.e. the frame $\mathcal{O}(X)$ is generated by its complemented elements) and every open covering of X has a pairwise-disjoint refinement.

Every zero-dimensional locale which is Lindelöf (in particular, every compact zero-dimensional locale) is strongly zero-dimensional, since if we are given a countable cover of X by clopens U_i $(i \geq 0)$, we can refine it to the cover by clopens $V_i = U_i \setminus (\bigcup_{j=0}^{i-1} U_j)$. Also, every locale X such that $\mathcal{O}(X)$ is a complete Boolean algebra is strongly zero-dimensional (cf. [6, 5.39]). However, not every zero-dimensional locale is strongly zero-dimensional: a counterexample is given by the order topology on the first uncountable ordinal (cf. [17, Example 42]). The full subcategory of strongly zero-dimensional locales is reflective in the category of all locales; the unit of the adjunction is not in general a connected morphism or even a surjection (for a locale which is zero-dimensional but not strongly so, it is a non-trivial inclusion), but it is still the case that an arbitrary locale and its strongly zero-dimensional reflection induce the same monad on **Set**. (We shall not give an explicit proof of these assertions here, though in fact one could be extracted from the arguments which follow.)

Now if X is a zero-dimensional locale, then its frame $\mathcal{O}(X)$ may be represented as the frame of C-ideals for a suitable coverage C (in the sense of [8, II 2.11]) on the Boolean algebra of clopen sublocales of X (= complemented elements of $\mathcal{O}(X)$). In terms of the geometric morphism $f : \mathbf{Sh}(X) \to \mathbf{Set}$, the latter may be identified with the elements of $f_*f^*(2)$ where 2 is the two-element set $\{0,1\}$, i.e. with the binary operations of the algebraic theory corresponding to the monad f_*f^*. This provides the justification for the construction which follows.

Reverting to the case of an arbitrary cartesian monad $\mathbb{T}$ on **Set**, we note that since T preserves finite products any algebraic structure carried by a set I is automatically inherited by TI; hence in particular $T2$ has a Boolean algebra structure. (Explicitly, if we take the top element to be the projection π_1 (where $\pi_1(x,y) = y$), then the binary meet operation is given by

$$(\alpha \wedge \beta)(x,y) = \alpha * \beta\left(\begin{smallmatrix} x & x \\ x & y \end{smallmatrix}\right) = \alpha(x, \beta(x,y)) ,$$

negation by $(\neg\alpha)(x,y) = \alpha(y,x)$, and so on. It is a straightforward exercise to verify using the hyperaffine identities that these operations satisfy the equations for Boolean algebras.)

With an arbitrary I-ary operation α of $\mathbb{T}$ (where I may be finite or infinite), we associate an I-tuple of binary operations $\alpha^{(i)}$: $\alpha^{(i)}(x,y)$ is α applied to the I-tuple with y in the ith place and x everywhere else. For example, if $I = 2$ then $\alpha^{(0)} = \neg\alpha$ and $\alpha^{(1)} = \alpha$.

Lemma 2.4. *As elements of $T2$, the $\alpha^{(i)}$ are pairwise disjoint, and their least upper bound is the top element π_1.*

Proof. If $i \neq j$, then $(\alpha^{(i)} \wedge \alpha^{(j)})(x,y)$ is obtained by applying $\alpha * \alpha$ to the matrix with y in position (i,j) and x everywhere else; by the hyperaffine identities this reduces to x, so $\alpha^{(i)} \wedge \alpha^{(j)} = \pi_0$. If β is a binary operation satisfying $\beta \geq \alpha^{(i)}$ for all i, then we have

$$\alpha(x, x, \ldots, x, \beta(x,y), x, \ldots, x) = \alpha(x, x, \ldots, x, y, x, \ldots, x)$$

(where the entries other than x occur in the ith place on each side) for all $i \in I$; so if we apply $\alpha * \alpha$ to the two matrices which have $\beta(x,y)$ (resp. y) on the main diagonal and x everywhere else, we obtain the same result. But the first of these reduces to $\beta(x,y)$ and the second to y; so β must be the top element π_1 of $T2$. $\square$

More generally, for any subset I' of the arity I of α we may define $\alpha^{(I')}$ to be the binary operation obtained by applying α to the I-tuple whose ith member is y if $i \in I'$, and x otherwise. Then an argument like that just given shows that, for each I', $\alpha^{(I')}$ is the least upper bound in $T2$ of the set $\{\alpha^{(i)} \mid i \in I'\}$ (and, in particular, all such sets have least upper bounds in $T2$, although $T2$ is not in general complete).

We define a coverage C on $T2$ by taking $C(\beta)$, for a binary operation β, to consist of all sets of the form $\{\alpha^{(i)} \mid i \in I'\}$, as (α, I') ranges over all pairs consisting of an operation α (of arbitrary arity) of $\mathbb{T}$ and a subset I' of its arity such that $\beta = \alpha^{(I')}$. In particular, we note that the bottom element π_0 of $T2$ is covered by the empty family, since it equals $\alpha^{(\emptyset)}$ for any α. To verify that C satisfies the pullback-stability property of [8, II 2.11], suppose $\gamma \leq \beta = \alpha^{(I')}$. Then if we define δ to be the $I+1$-ary operation given by

$$\delta(x_0, x_i \mid i \in I) = \gamma(x_0, \alpha(x_i \mid i \in I))$$

(where 0 is a subscript not in I), it is easily seen that $\delta^{(i)} = \gamma \wedge \alpha^{(i)}$ for each $i \in I$, and hence that $\delta^{(I')} = \gamma \wedge \alpha^{(I')} = \gamma$; so $\{\gamma \wedge \alpha^{(i)} \mid i \in I'\} \in C(\gamma)$, as required.

By the remark about binary operations before Lemma 2.4, we know that $\{\alpha, \neg\alpha\}$ is in $C(\pi_1)$ for all $\alpha \in T2$, and hence that $C(\beta)$ contains all disjoint pairs (γ, δ) whose join is β. It follows that any C-ideal of $T2$ is an ideal in the usual sense; on the other hand, any principal ideal is a C-ideal, by the remark following Lemma 2.4. We shall also require

Lemma 2.5. *The coverage C has local character: that is, given $R \in C(\beta)$ and $S_\gamma \in C(\gamma)$ for each $\gamma \in R$, the set $\bigcup_{\gamma \in R} S_\gamma$ is in $C(\beta)$.*

Proof. For simplicity, we shall assume that β is the top element π_1; so we can take R to be $\{\alpha^{(i)} \mid i \in I\}$ for some I-ary operation α. And each $S_{\alpha^{(i)}}$ has the form

$$\{\delta_i^{(j)} \mid j \in J_i'\}$$

for some J_i-ary operation δ_i and some subset $J_i' \subseteq J_i$ such that $\alpha^{(i)} = \delta^{(J_i')}$; for convenience, we shall assume that the index sets J_i are disjoint. Let $K = \bigcup_{i \in I} J_i$, and consider the K-ary operation ζ which acts on a K-tuple of variables by first applying δ_i to those variables whose indices lie in J_i, for each i, and then applying α to the resulting I-tuple. It is easy to see that we have $\zeta^{(j)} = \alpha^{(i)} \wedge \delta_i^{(j)}$ for each $j \in J_i$ and each i; in particular, $\zeta^{(j)} = \pi_0$ if j lies in $J_i \setminus J_i'$ for some i, from which we may deduce that ζ does not depend on its jth variable for any such j. Hence, by Lemma 2.2, ζ is the image under the inclusion of an operation ζ' of arity $K' = \bigcup_{i \in I} J_i'$, which satisfies $\zeta'^{(j)} = \delta_i^{(j)}$ for all $j \in J_i'$, and thus witnesses the fact that $\bigcup_{i \in I} S_{\alpha^{(i)}}$ is in $C(\pi_1)$. □

It follows that the C-ideal generated by an arbitrary downwards-closed subset S of $T2$ may be constructed in a single step: it consists of all those $\beta \in T2$ for which there exists $R \in C(\beta)$ such that $R \subseteq S$.

Corollary 2.6. *Let $(S_i \mid i \in I)$ be a pairwise-disjoint family of elements of the frame C-Idl$(T2)$ of C-ideals of $T2$, whose join is the top element. Then each S_i is a principal ideal, and there exists an I-ary operation α of $\mathbb{T}$ such that $S_i = (\alpha^{(i)})$ for each i.*

Proof. The join of the S_i in C-Idl$(T2)$ is the C-ideal generated by their union; so by the remark above we have an operation γ of $\mathbb{T}$ (whose arity J will, in general, be different from I) such that $\gamma^{(j)} \in \bigcup_{i\in I} S_i$ for all j. Further, we may assume by Lemma 2.2 that γ depends on all its variables (i.e. it is not in the image of $TJ' \rightarrowtail TJ$ for any proper subset J' of J), and so since $S_i \cap S_{i'}$ is the least element $\{\pi_0\}$ of C-Idl$(T2)$ for all $i \neq i'$, it follows that for each j there is a *unique* i such that $\gamma^{(j)} \in S_i$. Let J_i be the set of indices j for which $\gamma^{(j)} \in S_i$, and let $p : J \to I$ be the projection sending each j to the unique i such that $j \in J_i$. Now let $\alpha = Tp(\gamma)$ be the I-ary operation obtained by applying γ to the J-tuple $(x_{p(j)} \mid j \in J)$. Then it is clear that $\alpha^{(i)} = \gamma^{(J_i)}$ for each i. But $\gamma^{(J_i)}$ is covered by the set $\{\gamma^{(j)} \mid j \in J_i\}$, which is contained in S_i; so $\gamma^{(J_i)} \in S_i$. A similar argument shows that $\neg\gamma^{(J_i)}$ belongs to the join $\bigvee_{i'\neq i} S_{i'}$, from which it follows that $\gamma^{(J_i)}$ must be the largest element of S_i. □

We are now ready for the main result of this section.

Theorem 2.7. *Let $\mathbb{T} = (T, \eta, \mu)$ be a cartesian monad on* **Set**, *and let X be the locale corresponding to the frame of C-ideals of the Boolean algebra $T2$, as defined above. Then X is strongly zero-dimensional, and the monad on* **Set** *induced by the unique geometric morphism $f : \mathbf{Sh}(X) \to \mathbf{Set}$ is isomorphic to $\mathbb{T}$.*

Proof. For the zero-dimensionality, we note that the principal ideals are complemented in C-Idl$(T2)$, and every C-ideal is a union of principal ideals. For strong zero-dimensionality, suppose we have a (not necessarily disjoint) family of C-ideals $(S_i \mid i \in I)$ whose join is the whole of $T2$. Then it is still true that we can find a J-ary operation γ, for some J, such that each $\gamma^{(j)}$ belongs to at least one S_i; the latter will no longer be unique, but by making choices we can partition the index set J into subsets J_i as before, and hence construct a family of elements $(\beta_i \mid i \in I)$ such that $\beta_i \in S_i$ for each i, and the principal ideals (β_i) are pairwise disjoint and have join $T2$. So we have refined our given covering to a pairwise-disjoint one.

For the second assertion, we note that elements of $f_*f^*(I)$, for any set I, may be identified with global sections of the constant sheaf $f^*(I)$; equivalently, they correspond to locale maps from X to the discrete locale I, or equivalently again to I-indexed pairwise-disjoint open coverings of X. But each $\alpha \in TI$ gives rise to such a covering, by the principal ideals $(\alpha^{(i)})$; and we saw in Corollary 2.6 that every pairwise-disjoint covering arises in this way. Furthermore, if β is another i-ary operation such that $\alpha^{(i)} = \beta^{(i)}$ for all i, we see that $\alpha * \alpha$ and $\beta * \alpha$ agree when applied to the $(I \times I)$ matrix whose diagonal entries are $(x_i \mid i \in I)$ and whose off-diagonal entries are all taken to be some other variable y; but $\beta * \alpha = \beta \dagger \alpha$ since $\mathbb{T}$ is commutative, and $\beta \dagger \alpha$ similarly agrees with $\beta \dagger \beta$ on this matrix. Hence $\alpha = \beta$. Thus we have established a bijection from TI to $f_*f^*(I)$; it is straightforward to verify that this bijection is natural in I, and that it is a morphism of monads. □

Remark 2.8. In particular, Theorem 2.7 provides the converse to Lemma 1.3, which was promised earlier. As an unexpected bonus, it also provides a proof that any cartesian functor $T : \mathbf{Set} \to \mathbf{Set}$ which carries a monad structure has a rank (that is, it preserves κ-filtered colimits for some cardinal κ – it suffices to take κ so large that any pairwise-disjoint family of elements of $T2$ has fewer than κ nonzero members). It will be recalled that, in [2], Blass showed that the answer to the question 'Does every cartesian functor $\mathbf{Set} \to \mathbf{Set}$ have a rank?' depends on set-theoretic assumptions about measurable cardinals; but we do not need to invoke these in the case of functors with monad structures.

We note that, if the functor T has rank κ, it suffices to consider only the operations of arity less than κ in defining the coverage C: in particular if T is finitary (i.e. has rank ω), then $C(\beta)$ consists precisely of the finite pairwise-disjoint families in $T2$ with join β. Thus, the C-ideals in this case are exactly the ideals of $T2$ in the usual sense; in other words, the locale X is just the Stone space of the Boolean algebra $T2$ (cf. [8, II 4.4]). As we remarked in [10], the finitary algebraic theory which corresponds in this way to a Boolean algebra B is just the 'hyperaffine part' of the theory of B-modules; equivalently, it is the theory of actions of B as considered by Bergman [1].

In the opposite direction to Theorem 2.7, we have:

Lemma 2.9. *Let X be a strongly zero-dimensional locale, and let $\mathbb{T}$ be the cartesian monad on* **Set** *induced by the geometric morphism* $\mathbf{Sh}(X) \to \mathbf{Set}$. *Then X is isomorphic to the locale constructed from $\mathbb{T}$ as in Theorem* 2.7.

Proof. Since X is zero-dimensional, we can represent the corresponding frame $\mathcal{O}(X)$ as the frame of C-ideals of its sublattice B of complemented elements, where C is the coverage such that $C(b)$ consists of all those subsets of B whose join in $\mathcal{O}(X)$ is b. Further, since X is strongly zero-dimensional, we may replace C by the coverage C' consisting of all pairwise-disjoint covers in C. But we know that the elements of B (that is, the clopen sublocales of X) may be identified with elements of $T2$; and similarly each pairwise-disjoint cover of X by clopen sublocales defines an element of TI, where I is the set indexing the cover. So it is straightforward to identify the coverage C' with the one constructed from $\mathbb{T}$ as above. □

Thus, the cartesian monads on **Set** correspond bijectively to strongly zero-dimensional locales (cf. [10, 8.8]). Indeed, the correspondence may be made functorial. If $f : X \to Y$ is a continuous map between strongly zero-dimensional locales, then f^* preserves complemented elements of their open-set lattices, and so yields a map $S2 \to T2$, where $\mathbb{S}$ and $\mathbb{T}$ are the cartesian monads corresponding to Y and X, respectively. Further, since elements of SI correspond to I-indexed pairwise-disjoint families of elements of $S2$, and these are preserved by f^*, we actually have maps $SI \to TI$ for all I; it is not hard to see that these form a natural transformation, and indeed a morphism of monads $\mathbb{S} \to \mathbb{T}$. In the converse direction, a morphism of monads $\mathbb{S} \to \mathbb{T}$ yields in particular a Boolean algebra homomorphism $S2 \to T2$, which is readily seen to be 'continuous' for the coverages constructed as above, and hence to yield a frame homomorphism $C\text{-Idl}(S2) \to C\text{-Idl}(T2)$. We have thus established

Theorem 2.10. *The category of cartesian monads on* **Set** *(and arbitrary monad morphisms between them) is dual to the category of strongly zero-dimensional locales (and arbitrary continuous maps between them).*

Extending the above ideas, we may (as hinted earlier) construct the strongly zero-dimensional reflection of an arbitrary locale X by first forming the monad $\mathbb{T}$ induced by the geometric morphism $\mathbf{Sh}(X) \to \mathbf{Set}$, and then forming the strongly zero-dimensional locale which corresponds to $\mathbb{T}$. We shall not pursue the matter here.

In this section, we have not discussed the relationship between the topos $\mathbf{Sh}(X)$ constructed from a cartesian monad $\mathbb{T}$ on **Set**, and the category $\mathbf{Set}^{\mathbb{T}}$ of algebras for $\mathbb{T}$. However, using the results of [10], the relationship is not hard to determine. Since $\mathbf{Sh}(X)$ has coequalizers, the comparison functor $\mathbf{Sh}(X) \to \mathbf{Set}^{\mathbb{T}}$ has a left adjoint; this left adjoint is full and faithful, and so $\mathbf{Set}^{\mathbb{T}}$ may be identified with a coreflective subcategory of $\mathbf{Sh}(X)$, which consists of the well-supported objects of this topos together with 0 – in other words, it is the two-valued collapse of $\mathbf{Sh}(X)$ as defined in [10]. So the topos $\mathbf{Sh}(X)$ could alternatively have been constructed from $\mathbf{Set}^{\mathbb{T}}$ by the 'Reconstruction Theorem' 6.1 of [10].

3. Strong functors and indexed functors

We recall that, if $(\mathscr{E}, \otimes, U)$ is a symmetric monoidal category, a (*tensorial*) *strength* on a functor $T : \mathscr{E} \to \mathscr{E}$ is a natural transformation $\tau_{I,J} : I \otimes TJ \to T(I \otimes J)$ such that the diagrams

$$\begin{array}{ccc} TI & \longrightarrow & U \otimes TI \\ & \searrow & \downarrow{\scriptstyle \tau_{U,I}} \\ & & T(U \otimes I) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} I \otimes (J \otimes TK) & \xrightarrow{I \otimes \tau_{J,K}} & I \otimes T(J \otimes K) \\ \downarrow & & \downarrow{\scriptstyle \tau_{I,J \otimes K}} \\ (I \otimes J) \otimes TK & \xrightarrow{\tau_{I \otimes J,K}} & T(I \otimes J \otimes K) \end{array}$$

commute, where the unlabelled arrows are canonical isomorphisms of the monoidal structure. It is well known (cf. [13]) that, if the monoidal structure is closed, specifying a strength on T is equivalent to specifying an $\mathscr{E}$-enrichment of T, where $\mathscr{E}$ is regarded as enriched over itself in the usual way. In this paper we shall be exclusively concerned with the case where the monoidal structure is that given by finite products in $\mathscr{E}$.

In general, a functor $T : \mathscr{E} \to \mathscr{E}$ may not admit a strength (for example, the functor $\widetilde{T}$ considered in Lemma 1.5 does not, except in degenerate cases), or it may admit several different ones. Similarly, a natural transformation $\alpha : S \to T$ between (the underlying ordinary functors of) two strong functors may or may not be a *strong natural transformation*, in the sense that the diagram

$$\begin{array}{ccc} I\otimes SJ & \xrightarrow{\sigma_{I,J}} & S(I\otimes J) \\ {\scriptstyle I\otimes\alpha_J}\downarrow & & \downarrow{\scriptstyle \alpha_{I\otimes J}} \\ I\otimes TJ & \xrightarrow{\tau_{I,J}} & T(I\otimes J) \end{array}$$

commutes for all I and J. However, there is an important special case in which the strength exists and is unique in a suitable sense: the following result is essentially due to Kock [12], but we give the proof here for completeness.

Proposition 3.1. *Let $\mathscr{E}$ be a category with finite products, and $T : \mathscr{E} \to \mathscr{E}$ an (ordinary) functor which preserves finite products. Suppose further that there is a natural transformation $\eta : 1_{\mathscr{E}} \to T$. Then there is a unique strength on T making η into a strong natural transformation. If η is the unit of a monad structure $\mathbb{T} = (T, \eta, \mu)$, then this strength also makes μ a strong natural transformation.*

Proof. We define $\tau_{I,J}$ to be the composite

$$I \times TJ \xrightarrow{\eta_I \times 1_{TJ}} TI \times TJ \cong T(I \times J);$$

it is clear that this is natural in I and J, and that it satisfies the first condition of the definition. For the second, we need to observe that

$$\begin{array}{ccc} I\times J & \xrightarrow{\eta_{I\times J}} & T(I\times J) \\ & {\scriptstyle \eta_I\times\eta_J}\searrow & \downarrow \\ & & TI\times TJ \end{array}$$

commutes; but this is immediate from the naturality of η, since the vertical isomorphism is obtained by applying T to the two projections from $I \times J$. The same commutative diagram verifies that η is a strong natural transformation $1_{\mathscr{E}} \to T$. To verify that μ, if it exists, is also strong, we need the similar diagram identifying $\mu_{I\times J}$ with $\mu_I \times \mu_J$, plus the monad identity $\mu_I \circ T\eta_I = 1_{TI}$.

For the uniqueness, suppose $\sigma_{I,J} : I \times TJ \to T(I \times J)$ is any strength for T. Composing it with $T\pi_1$ and $T\pi_2$, we get natural transformations $\alpha_{I,J} : I \times TJ \to TI$ and $\beta_{I,J} : I \times TJ \to TJ$, which together determine σ. But $\beta_{I,J} = \beta_{1,J} \circ \pi_2$ by naturality, and $\beta_{1,J} = 1_{TJ}$ by the first condition in the definition of a strength. Similarly, $\alpha_{I,J} = \alpha_{I,1} \circ \pi_1$, and $\alpha_{I,1} = \eta_I$ by the fact that η is strong. So we have identified σ with the strength τ defined above. $\square$

Remark 3.2. For a functor which preserves 1, the possession of a natural transformation $1_{\mathscr{E}} \to T$ is a necessary condition for admitting a strength, since the components $\tau_{I,1}$ of any strength on T provide such a transformation (which is actually the unique strong natural transformation from the identity to T; cf. the remark after Lemma 1.3). Thus Proposition 3.1 in fact shows that, for a functor $T : \mathscr{E} \to \mathscr{E}$ which preserves finite products, the possible strengths on T are in bijective correspondence with the natural transformations $1_{\mathscr{E}} \to T$.

In topos theory we are accustomed to consider not only strong functors but also indexed functors: recall that if $\mathscr{E}$ is a cartesian category, an *$\mathscr{E}$-indexed functor* $T : \mathscr{E} \to \mathscr{E}$ consists of a family of functors $T^I : \mathscr{E}/I \to \mathscr{E}/I$, one for each object I of $\mathscr{E}$, commuting up to coherent isomorphism with the pullback functors $f^* : \mathscr{E}/J \to \mathscr{E}/I$ induced by morphisms $f : I \to J$ of $\mathscr{E}$. We identify $\mathscr{E}/1$ with $\mathscr{E}$, and regard T^1 as the 'underlying ordinary functor' of the indexed functor T. (We shall frequently abuse notation and denote T^1 simply by T.) *Indexed natural transformations* between indexed functors are defined in the obvious way.

The underlying ordinary functor of an indexed functor has a canonical strength. If I is an object of $\mathscr{E}$, we write $I^* : \mathscr{E} \to \mathscr{E}/I$ for the pullback functor along $I \to 1$, and Σ_I for its left adjoint (the forgetful functor). The diagonal morphism $I \to I \times I$ induces, for any J, a morphism $I^*(J) \to I^*(I \times J)$ in $\mathscr{E}/I$; applying T^I to this, we obtain a morphism $I^*T^1(J) \cong T^I I^*(J) \to T^I I^*(I \times J) \cong I^*T^1(I \times J)$, and transposing across the adjunction we obtain $\tau_{I,J} : I \times T^1 J \cong \Sigma_I I^* T^1(J) \to T^1(I \times J)$. We omit the (straightforward) verification that this is indeed a strength on T^1 (but cf. [15, p. 37]). Similarly, any indexed natural transformation $\alpha : S \to T$ gives rise to a strong natural transformation $\alpha^1 : S^1 \to T^1$.

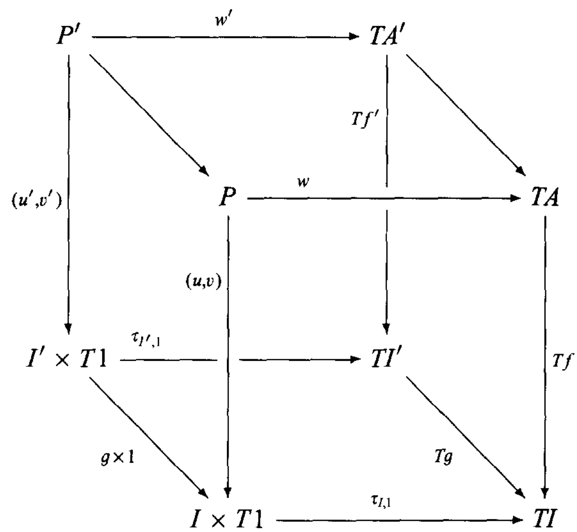
In the converse direction, we have the following result, which was proved by R. Paré around 20 years ago [14], but never published by him.

Proposition 3.3. *Let $\mathscr{E}$ be a cartesian category, and $T : \mathscr{E} \to \mathscr{E}$ a strong functor whose underlying ordinary functor preserves pullbacks. Then T extends to an $\mathscr{E}$-indexed functor. Moreover, this extension is unique (up to canonical isomorphism) if we require that it should preserve pullbacks as an indexed functor (i.e. that each T^I should preserve pullbacks), and induce the given strength on T. And if S is another such functor, then any strong natural transformation $\alpha : S \to T$ extends uniquely to an indexed natural transformation.*

Proof. Given an object $f : A \to I$ of $\mathscr{E}/I$, we define $T^I(f)$ to be the morphism $u : P \to I$, where

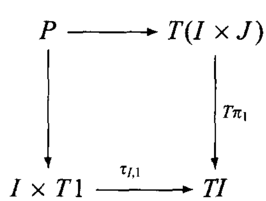
$$\begin{array}{ccc} P & \xrightarrow{\;w\;} & TA \\ \Big\downarrow{\scriptstyle (u,v)} & & \Big\downarrow{\scriptstyle Tf} \\ I \times T1 & \xrightarrow{\;\tau_{I,1}\;} & TI \end{array}$$

is a pullback square (and τ is the strength of T). Clearly, T^I is a functor $\mathscr{E}/I \to \mathscr{E}/I$. To show that the T^I form an indexed functor, let $g: I' \to I$ be a morphism of $\mathscr{E}$, and consider the cube



where f' is the pullback of f along g. Since T preserves pullbacks, the right vertical face of this cube is a pullback, and the front and back faces are pullbacks by definition. So the left face is a pullback, from which it follows that $u' = T^{I'}(f')$ is (isomorphic to) the pullback of u along g. A similar calculation shows that T^I preserves pullbacks as a functor $\mathscr{E}/I \to \mathscr{E}/I$; and since $\tau_{1,1}$ is the canonical isomorphism $\pi_2 : 1 \times T1 \to T1$, it is easy to see that T^1 is (isomorphic to) our original functor T.

Next, we must show that the strength (τ', say) induced on T^1 by the indexing is the one originally given. The description before the statement of the Proposition tells us that, in order to construct $\tau'_{I,J}$, we should compose the top edge of the pullback square



with the isomorphism $P \cong \Sigma_I I^*(TJ) \cong I \times TJ$. But the instance of the commutative cube above with $I' = I$ and $I = 1$, plus the fact that $\tau_{1,1}$ is the identity, yields the fact that the naturality squares

$$\begin{array}{ccc} I \times TJ & \xrightarrow{\tau_{I,J}} & T(I \times J) \\ \downarrow & & \downarrow \\ I \times T1 & \xrightarrow{\tau_{I,1}} & TI \end{array}$$

for τ are in fact pullbacks; so this construction simply yields $\tau_{I,J}$.

For the uniqueness, we observe that for any $f : A \to I$ we have a pullback square

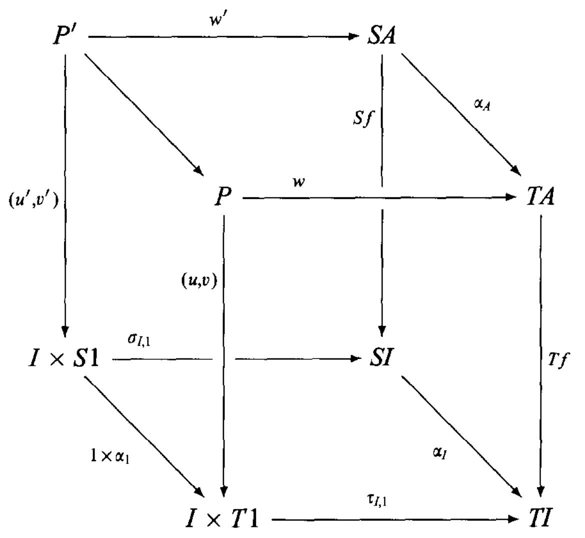
$$\begin{array}{ccc} A & \xrightarrow{(1,f)} & A \times I \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \times 1} \\ I & \xrightarrow{\Delta} & I \times I \end{array}$$

which we may regard as a pullback

$$\begin{array}{ccc} f & \longrightarrow & I^*A \\ \downarrow & & \downarrow{\scriptstyle I^* f} \\ I^*1 & \xrightarrow{\Delta} & I^*I \end{array}$$

in $\mathscr{E}/I$. So, if we are given that T^I preserves pullbacks and that $T^I I^*$ is isomorphic to I^*T^1, the effect of T^I will be determined up to canonical isomorphism once we know what it does to the morphism $\Delta : I^*1 \to I^*I$. But the latter is forced upon us by the strength: $T^I(\Delta) : I^*T^1(1) \to I^*T^1(I)$ must be the transpose of $\tau_{I,1} : \Sigma_I I^* T^1(1) \cong I \times T^1(1) T^1(I)$.

Finally, suppose $\alpha : S \to T$ is a strong natural transformation. Then for any $f : A \to I$, we have a unique morphism between the pullbacks P' and P making the cube



commute. It is straightforward to verify that these morphisms form a natural transformation $\alpha^I : S^I \to T^I$, and that they fit together to form an indexed natural transformation. The uniqueness is proved by an argument similar to that in the last paragraph. □

Remark 3.4. The author is indebted to Edmund Robinson for the observation (apparently originally due to Gordon Plotkin) that, if we index the category $\mathscr{E}$ over itself by taking the fibre $\mathscr{E}^I$ to be not the whole of $\mathscr{E}/I$ but its full subcategory whose objects are of the form I^*A for some object A of $\mathscr{E}$ (that is, if we replace the Eilenberg–Moore category of the comonad $(-) \times I$ by the Kleisli category of the same comonad), then specifying a strength for a functor $T : \mathscr{E} \to \mathscr{E}$ is precisely equivalent to specifying an extension of T to an $\mathscr{E}$-indexed functor. It is conceivable that a simpler proof of Paré's result could be given by combining this observation with standard results on the lifting of functors to Eilenberg–Moore categories, but as yet I have not been able to find such a proof.

Combining Propositions 3.1 and 3.3, we obtain:

Corollary 3.5. *Let $\mathscr{E}$ be a cartesian category. Then any cartesian monad on $\mathscr{E}$ has a canonical extension to an indexed cartesian monad.*

Proof. The only point which requires further comment is the fact that the indexed extension of T preserves all finite limits, and not just pullbacks. But this follows from the fact that T^1 preserves the terminal object and the T^I form an indexed functor. □

Remark 3.6. In the case when $\mathscr{E}$ is a topos, and the monad $\mathbb{T}$ is induced by a geometric morphism $f : \mathscr{F} \to \mathscr{E}$, we do not need to invoke Corollary 3.5: for it is well known that any topos defined over $\mathscr{E}$ becomes an $\mathscr{E}$-indexed category by setting $\mathscr{F}^I = \mathscr{F}/f^*I$, and the adjunction $(f^* \dashv f_*)$ lifts to an $\mathscr{E}$-indexed adjunction, so that it induces an $\mathscr{E}$-indexed monad.

4. Cartesian monads on a general topos

In this section, we shall assume given a topos $\mathscr{E}$ equipped with a monad $\mathbb{T} = (T, \eta, \mu)$ such that T is cartesian. By the results of the last section, $\mathbb{T}$ has a canonical extension to an indexed monad, and we shall exploit this by arguing informally in the internal logic of $\mathscr{E}$: when we speak of families of entities indexed by an object I of $\mathscr{E}$, we of course mean entities of the appropriate type in $\mathscr{E}/I$.

In the first place, since $\mathbb{T}$ is strong, we may interpret elements of TI as I-ary operations on $\mathbb{T}$-algebras, as in the classical case $\mathscr{E} = \mathbf{Set}$: specifically, for any $\mathbb{T}$-algebra (A, α) we have a map

$$TI \times A^I \cong A^I \times TI \xrightarrow{\tau_{A^I,I}} T(A^I \times I) \xrightarrow{T(\mathrm{ev})} TA \xrightarrow{\quad\alpha\quad} A$$

equipping A with a TI-indexed family of I-ary operations, for each object I of $\mathscr{E}$. (Indeed, it will be recalled that in [11] the notion of $\mathscr{E}$-indexed monad on $\mathscr{E}$ was taken as the definition of an algebraic theory 'internal to $\mathscr{E}$'.) Of course, we say that an operation $A^I \to A$ is *hyperaffine* if it satisfies the two commutative diagrams (2); the proof of Theorem 2.1 is entirely constructive, and so we know that in our case all the operations induced by elements of TI are hyperaffine.

A reader having some familiarity with the internal logic of a topos should have no difficulty in re-interpreting the remaining proofs from Section 2 in this context, provided the topos $\mathscr{E}$ is Boolean. We may thus conclude:

Theorem 4.1. *If $\mathscr{E}$ is a Boolean topos, then every cartesian monad on $\mathscr{E}$ is induced by a geometric morphism $f : \mathscr{F} \to \mathscr{E}$. Moreover, the category of cartesian monads on $\mathscr{E}$ is dual to the category of strongly zero-dimensional internal locales in $\mathscr{E}$.*

However, for a non-Boolean $\mathscr{E}$, the argument begins to break down at Lemma 2.2. If α is an element of TI, then the subobjects $I' \rightarrowtail I$ such that α is in the image of $TI' \rightarrowtail TI$ still form a filter F in Ω^I; but, whilst we can still represent the intersection I_0 of all the members of F as an I-indexed intersection of 'co-singletons' $S_i = \{i' \in I \mid (i' = i) \Rightarrow (i' \in I_0)\}$, the latter need not belong to F in general, and so we cannot apply the argument in the proof of Lemma 2.2. (To prove that the S_i belong to F would require the information that all such 'co-singletons' are 'co-compact' in Ω^I; but this would imply that Ω^I was a co-Heyting-algebra, which is well known to be equivalent to Booleanness of $\mathscr{E}$; cf. [16].) Indeed, this failure is not surprising; for if $\mathbb{T}$ is the monad induced by a geometric morphism $f : \mathscr{F} \to \mathscr{E}$, there is no reason why f^* should preserve arbitrary intersections of subobjects (unless $\mathscr{E}$ is Boolean, in which case the result follows from the fact that f^* preserves complements and arbitrary unions).

In the non-Boolean case, when we seek to construct an internal locale X in $\mathscr{E}$ such that the geometric morphism $\mathbf{Sh}_{\mathscr{E}}(X) \to \mathscr{E}$ induces the monad $\mathbb{T}$, we must replace the Boolean algebra $T2$ considered in Section 2 by the object $T\Omega$ of Ω-ary operations of $\mathbb{T}$. Of course, this object has a natural Heyting algebra structure, and we should seek to define a coverage on it with properties similar to those of the coverage C considered in Section 2. We note that, as before, each element of TI (for I an arbitrary object of $\mathscr{E}$) induces an 'I-tuple of Ω-ary operations': that is, we have a map $TI \to (T\Omega)^I$, whose transpose is the composite

$$I \times TI \xrightarrow{\tau_{I,I}} T(I \times I) \xrightarrow{T\delta} T\Omega,$$

where $\delta : I \times I \to \Omega$ classifies the diagonal subobject $I \rightarrowtail I \times I$. Informally, we may denote this map by $\alpha \mapsto (\alpha^{(i)} \mid i \in I)$, where

$$\alpha^{(i)}(x_p \mid p \in \Omega) = \alpha(x_{[\![i=i']\!]} \mid i' \in I).$$

Similarly, we may write $\alpha^{(I')}$, where I' is a subobject of I, for the Ω-ary operation defined by

$$\alpha^{(I')}(x_p \mid p \in \Omega) = \alpha(x_{[\![i \in I']\!]} \mid i \in I),$$

i.e. for the I'th component of the image of α under the map $TI \to T\Omega^{\Omega^I}$ whose transpose is

$$\Omega^I \times TI \xrightarrow{\tau_{\Omega^I,I}} T(\Omega^I \times I) \xrightarrow{T(\mathrm{ev})} T\Omega.$$

It is no longer possible to talk about the family $(\alpha^{(i)} \mid i \in I)$ being 'pairwise disjoint', as we did in Section 2; but we have a substitute for this: namely, for any two subobjects I', I'' of I, we have

$$\alpha^{(I')} \wedge \alpha^{(I'')} = \alpha^{(I' \cap I'')}.$$

For, by definition, $\alpha^{(I')} \wedge \alpha^{(I'')}$ is obtained by applying $\alpha * \alpha$ to the $(I \times I)$ matrix whose (i,j)th entry is $x_{[\![(i \in I') \wedge (j \in I'')]\!]}$. And if we apply α^d to this matrix, we obtain $\alpha^{(I' \cap I'')}$. If we could also prove that $\alpha^{(I')} = \bigvee\{\alpha^{(i)} \mid i \in I'\}$ for any $I' \rightarrowtail I$, then we would know that $I' \mapsto \alpha^{(I')}$ is a 'frame homomorphism' $\Omega^I \to T\Omega$ (though of course $T\Omega$ is not in general complete).

However, it is not clear whether this result holds constructively: the proof given in Lemma 2.4 for the case when I' is the whole of I is constructive, but the generalization mentioned after that proof appears to require the ability to distinguish between elements of I which lie in I' and those which do not – that is, it requires I' to be complemented in I. Nevertheless, it seems possible that the result might still be true for arbitrary I'; if it were, then we could define a coverage C on $T\Omega$, as we did in Section 2. Of course, this definition formally involves a proper class of covers, and so cannot be 'internalized' as it stands. However, the key to getting round this difficulty is contained in Remark 2.8, where we observed that we could in fact bound the arities of the operations used to generate covers; the analogue in our more general situation is the observation that, if α is an element of TI, then for each $i \in I$ we have $\alpha^{(i)} = \alpha^{(I')}$ where $I' = \{i' \in I \mid \alpha^{(i)} = \alpha^{(i')}\}$, and hence the cover generated by α is the same as that generated by the J-ary operation $Tq(\alpha)$, where $I \overset{q}{\twoheadrightarrow} J \rightarrowtail T\Omega$ is the image factorization of the map $i \mapsto \alpha^{(i)}$. Thus we may restrict our attention to operations whose arities are subobjects of $T\Omega$; and these can of course be indexed by an object of our topos $\mathscr{E}$ (specifically, by $\Sigma_{\Omega^{T\Omega}} T^{\Omega^{T\Omega}}(\in_{T\Omega} \rightarrow \Omega^{T\Omega})$, where $\in_{T\Omega} \rightarrowtail \Omega^{T\Omega} \times T\Omega$ is the subobject classified by the evaluation map).

The proof in Lemma 2.5 that the coverage has local character can be made constructive without much difficulty. The constructive analogue of Corollary 2.6 would then be a result asserting that any frame homomorphism $\Omega^I \rightarrow C\text{-Idl}(T\Omega)$ (where I is an arbitrary object of $\mathscr{E}$) takes principal ideals as values (note that all principal ideals would be C-ideals if the analogue of Lemma 2.4 were true), and is of the form $I' \mapsto (\alpha^{(I')})$ for some $\alpha \in TI$. If such a result were true, we could then readily conclude as in Theorem 2.7 that the geometric morphism $\mathbf{Sh}_{\mathscr{E}}(X) \rightarrow \mathscr{E}$ induces the monad $\mathbb{T}$, where X is the internal locale in $\mathscr{E}$ corresponding to the frame $C\text{-Idl}(T\Omega)$. However, there seem to be further problems at this point, since our proof of Corollary 2.6 makes use of Lemma 2.2 (which we have already seen to be constructively false); without this, we should obtain not an I-ary operation but a J-ary one for some J mapping epimorphically to I – and we have no reason to suppose that the functor T preserves epimorphisms.

At present, I am unable to find any way round these difficulties; but I have not abandoned hope that it might be possible to prove along these lines that any cartesian monad on a topos has a Jibladze factorization, and hence that its category of algebras is a collapsed topos.

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